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Maysel's formula of thermoelasticity extended to anisotropic materials at finite strain

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Abstract

The present paper is devoted to an extension of Maysel's formula from the linear theory of thermoelasticity to the geometrically non-linear theory of anisotropic solids and structures. The material description of continuum mechanics is used, and the constitutive equations of St. Venant and Kirchhoff are considered under two circumstances. First, we model the thermally induced deformation of anisotropic solids from an undistorted reference state. This leads to a non-linear integral equation for the thermal deformation, extending Maysel's formula of the infinitesimal theory to the regime of moderately large strains. Secondly, the linearized form of the constitutive relation is used to describe an infinitesimal strain superimposed upon a given, intermediate state of stress in a hyperelastic material of arbitrary type. The intermediate strain needs not be small. It is shown that, when properly interpreted, Maysel's formula may be applied directly to this second case, without any additional terms. The presented results should be of a rather general interest, since they lead to an efficient and powerful representation of the thermal response of anisotropic materials. As a structural application, Maysel's formula is subsequently derived for shear-deformable beams made of a St. Venant–Kirchhoff material. Geometric non-linearity is taken into account according to the assumptions of v. Karman, and the influence of shear is considered in the sense of Timoshenko. A semi-analytic solution procedure is derived for the case of simply supported beams with fixed ends. The thermally induced deflection is derived in closed form, and a non-linear equation is presented for the corresponding normal force. In the post-buckling regime, three branches of the solution are found in the considered range of thermal loading. Thermally loaded beams made of pyrolitic–graphite type material are studied, and the strong influence of the characteristic parameter of anisotropy is demonstrated. Stability of the solution is discussed, and results are compared to finite element computations in the isotropic case. An excellent agreement is found, and a paradoxical behavior of the finite element code ABAQUS is clarified. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Maysel's formula; Finite deformation theory; Timoshenko beam; v. Karman type of non-linearity

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1. Introduction

Maysel's formula of the linearized theory of thermoelasticity renders the thermoelastic displacement by volume integration (Maysel, 1941). The volume integral derived by Maysel (1941) contains influence functions for isothermal stresses as its kernel, which is multiplied by the temperature distribution. The value of Maysel's formula thus lies in the fact that known solutions of isothermal problems can be utilized for deriving thermoelastic solutions. The isothermal stresses occurring as kernels in Maysel's formula are attributed to dummy forces.

Various applications of Maysel's formula have been presented in Nowacki's monograph (Nowacki, 1962), establishing the popularity of Maysel's formula. The formula has been stated in the context of the Betti-Maxwell theorem by Kovalenko (1969) and by Carlson (1972). A derivation by means of the principle of virtual work has been presented in the monograph of Parkus (1976), where a complementary dummy load problem has been introduced in addition to the thermal stress problem, and the virtual work principle has been applied to both problems. Different boundary conditions in these two states of equilibrium have been taken into account in the state-of-the-art report of Ziegler and Irschik (1987). The present paper is devoted to an extension of these linear formulations to the geometrically non-linear theory of thermally loaded anisotropic solids and structures.

In order to give a short overview, the arrangement of our derivations is shortly described in the following. Finite deformations are taken into account by means of the material description of continuum mechanics (see Truesdell and Noll, 1972, Section 43A). Two equilibrium configurations are considered, where a common reference configuration is used. One of the two equilibrium configurations represents a solution of the thermal stress problem under consideration, and the second one corresponds to a dummy load problem. The boundary conditions of these two problems need not to be the same. Of course, one (or both) of these solutions may correspond to the infinitesimal theory. Coupling between deformation and temperature may or may not be taken into account in the procedure. The (true) field of displacements corresponding to one of the two equilibrium configurations is used as a virtual displacement (test function) in the virtual work statement for the second one, and vice versa. We allow to use displacement fields with respect to some stressed intermediate configuration, which again is taken to be common to both problems. A special hyperelastic relation, called the St. Venant–Kirchhoff material, is considered (see Truesdell and Noll, 1972, Section 82A and 94). The St. Venant–Kirchhoff constitutive relation represents an extension of the standard linear thermoelastic law of the infinitesimal theory by replacing the stress by the second Piola–Kirchhoff stress, and the infinitesimal strain by the Green strain. In the following, we apply the anisotropic version of the St. Venant–Kirchhoff constitutive equations under two circumstances. First, the intermediate configuration is assumed to coincide with the unstressed reference configuration. This situation is used to model the geometrically non-linear behavior of anisotropic solids and structures made of a St. Venant–Kirchhoff material. We then end up with a non-linear integral equation for the thermal deformation, extending Maysel's formula of the infinitesimal theory to the large strain regime. Secondly, the linearized form of the St. Venant–Kirchhoff constitutive equations is used to describe an infinitesimal strain superimposed upon a given, intermediate state of stress in a hyperelastic material of an arbitrary type. The intermediate strains may be large. It is shown that, when properly interpreted, the classical form of Maysel's formula may be applied directly in this second case, without any formal extension.

Despite the fact that the St. Venant–Kirchhoff constitutive model is known to be of a restricted practical use in the large strain regime, a number of theories for beams, plates and shells have been developed for the St. Venant–Kirchhoff material in order to take into account moderately large deviations from the infinitesimal theory (see Truesdell and Noll, 1972, Section 94). We therefore represent a structural application of the above three-dimensional relations, where we derive Maysel's formula for shear-deformable beams made of an orthotropic St. Venant–Kirchhoff material. Geometric non-linearity is taken into account according to the assumptions of v. Karman (see Rothert and Gensichen, 1987), and the influence of shear is considered in the sense of Timoshenko (see Ziegler, 1995). A semi-analytic solution procedure is presented for the case of simply supported beams with fixed ends. Using Maysel's formula, the thermally induced deflection is derived in closed form, and a non-linear equation is presented for the corresponding normal force occurring in the formula for the deflection. In the postbuckling regime, three branches of the solution are found in the considered range of thermal loading, two of them being stable. Thermally loaded beams made of pyrolitic–graphite type material are studied, and the strong influence of the characteristic parameter of anisotropy is demonstrated. Results are compared to finite element computations in the isotropic case. As a by-product, a paradoxical behavior of the finite element code ABAQUS is clarified in the example problem under consideration.

The following derivations are dedicated to an extension of the linear theory of thermoelasticity with respect to moderately large strains. A first account of our results for isotropic bodies has been presented by Holl et al. (1999). Moreover, the paper has been motivated by an analogy between thermal strains and incompatible parts of strain, such as viscoplastic or piezoelectric parts of strain. We hope to present corresponding results in the near future.

2. Principle of virtual work

Consider a body B in some current equilibrium configuration defined by its deformation map $\vec{x} = \phi(\vec{\xi})$ from an undistorted reference state. The place occupied by a material particle in the equilibrium position is denoted by $\vec{x}$, and $\vec{\xi}$ is the place of the particle in the reference configuration. In the material description, Cauchy's first law of motion of a particle assumes the form

$$\mathrm{Div}\tilde{P} + \vec{b}_0 = \vec{0}, \tag{2.1}$$

see Truesdell and Noll (1972, Section 43 A) for fundamentals of the material description. The subscript "0" recalls use of the reference configuration. The external body force is denoted by $\vec{b}_0$. "Div" is the divergence operator with respect to the place $\vec{\xi}$, and the tensor $\tilde{P}$ is the first Piola–Kirchhoff stress tensor, where

$$\tilde{P} = \tilde{F}\tilde{S}. \tag{2.2}$$

The (symmetric) second Piola–Kirchhoff stress tensor is denoted by $\tilde{S}$, and the deformation gradient tensor is

$$\tilde{F} = \nabla_{\xi}\vec{x} = \frac{\partial \vec{x}}{\partial \vec{\xi}}. \tag{2.3}$$

The deformation gradient tensor is related to the displacement gradient tensor by

$$\tilde{G}_u = \nabla_{\xi}\vec{u} = \frac{\partial \vec{u}}{\partial \vec{\xi}} = \tilde{F} - \tilde{I}, \tag{2.4}$$

where $\vec{u} = \vec{x} - \vec{\xi}$ is the displacement of the particle from the reference state. Cauchy's first law (Eq. (2.1)) now is multiplied by a virtual displacement field $\vec{\bar{u}}$. We note that displacements from an distorted intermediate configuration may be used as virtual displacements, without introducing problems into the material formulation. Integrating over the volume B_0 of the body in the reference configuration, applying the Gauss theorem and utilizing Cauchy's fundamental stress theorem, the following weak form of Eq. (2.1) is obtained:

$$\int_{\partial B_0} \vec{f}_0 \cdot \vec{\bar{u}}\,\mathrm{d}A_0 + \int_{B_0} \vec{b}_0 \cdot \vec{\bar{u}}\,\mathrm{d}V_0 - \int_{B_0} (\tilde{F}\tilde{S}) : \tilde{\bar{G}}_u\,\mathrm{d}V_0 = 0, \tag{2.5}$$

where the virtual displacement $\vec{\bar{u}}$ needs not be infinitesimal (see Marsden and Hughes, 1983). The surface of the body in the reference configuration is ∂B_0, and the Lagrange surface tractions are $\vec{f}_0$. The gradient of the virtual displacement $\vec{\bar{u}}$ with respect to the place $\vec{\xi}$ is denoted by

$$\tilde{\bar{G}}_u = \nabla_\xi \vec{\bar{u}} = \frac{\partial \vec{\bar{u}}}{\partial \vec{\xi}}, \tag{2.6}$$

(see Eq. (2.4)). For subsequent use, we note the definition of the Green strain tensor as

$$\tilde{E} = \tfrac{1}{2}(\tilde{G}_u + \tilde{G}_u^{\mathrm{T}} + \tilde{G}_u^{\mathrm{T}}\tilde{G}_u). \tag{2.7}$$

3. Pure thermal loading and dummy force problems

Let $T = T_0 + \theta$ be the actual temperature of the body, being at the uniform temperature T_0 in an intermediate equilibrium configuration. The 2. Piola–Kirchhoff stress in this intermediate configuration is denoted by $\tilde{S}_{\mathrm{i}}$, and $\tilde{F}_{\mathrm{i}}$ is the corresponding deformation gradient tensor with respect to the place $\vec{\xi}$ in the reference configuration. The index i refers to the intermediate configuration. It is assumed that the stress $\tilde{S}_{\mathrm{i}}$ is produced by some isothermal force loading. In case the intermediate configuration coincides with the unstressed reference configuration, we take $\tilde{S}_{\mathrm{i}} = \tilde{0}$ and $\tilde{F}_{\mathrm{i}} = \tilde{I}$. The change in temperature θ causes additional stresses $\tilde{S}_\theta$, where we assume no additional imposed body forces and imposed surface tractions to occur. The actual second Piola–Kirchhoff stress thus is $\tilde{S} = \tilde{S}_{\mathrm{i}} + \tilde{S}_\theta$. Thermally induced displacements relative to the intermediate configuration are denoted by $\vec{u}_\theta$, and $\tilde{G}_{u\theta}$ is the displacement gradient tensor of $\vec{u}_\theta$ with respect to the place $\vec{\xi}$ in the reference configuration. The actual deformation gradient tensor then is given by $\tilde{F} = \tilde{F}_{\mathrm{i}} + \tilde{G}_{u\theta}$. Note that this decomposition is additive, since we refer to the unstressed reference configuration for both, $\tilde{F}_{\mathrm{i}}$ and $\tilde{G}_{u\theta}$. The principle of virtual work (Eq. (2.5)) now reads

$$\int_{B_\theta} (\tilde{F}_{\mathrm{i}}\tilde{S}_\theta + \tilde{G}_{u\theta}\tilde{S}_{\mathrm{i}} + \tilde{G}_{u\theta}\tilde{S}_\theta) : \tilde{G}_{u\mathrm{d}}\,\mathrm{d}V_0 - \int_{\partial B_\theta} \vec{f}_{0\theta} \cdot \vec{u}_{\mathrm{d}}\,\mathrm{d}A_0 = 0. \tag{3.1}$$

∂B_θ denotes those parts of the surface, where the additional thermally induced reaction tractions $\vec{f}_{0\theta}$ are performing virtual work. In Eq. (3.1), the virtual displacement field $\vec{\bar{u}} = \vec{u}_{\mathrm{d}}$ represents the (true) displacements due to a system of dummy forces applied under isothermal conditions, i.e. without a change in temperature, $\theta_{\mathrm{d}}(\vec{\xi}) = 0$. The subscript d refers to the dummy forces. These dummy forces are assumed to be additional forces, applied to the intermediate equilibrium configuration of the body. The virtual displacements $\vec{u}_{\mathrm{d}}$ thus represent displacements relative to the intermediate configuration. The virtual deformations need not to be small. In case of single forces to be used as dummy loadings, the dummy system of forces may be written as

$$\vec{b}_{0\mathrm{d}}(\vec{\xi}) = \Sigma_{j=1}^{n} \vec{B}_j \delta(\vec{\xi} - \vec{\xi}_j), \tag{3.2}$$

where δ denotes the Dirac delta function, and $\vec{\xi}_j$ denotes the reference place of application of the load $\vec{B}_j$.

In Eq. (3.1), use has been made of the principle of virtual work applied to the intermediate state, and this relation has been subtracted from the work formulations for the final state. Hence, no external forces corresponding to the intermediate state do occur in Eq. (3.1). In using the (true) displacements of a certain equilibrium problem as virtual displacements for a second equilibrium problem, we follow an approach described by Truesdell and Noll (1972, Section 88) in the context of the Betti–Maxwell theorem for isothermal problems.

The body is assumed to be in equilibrium also under the action of the dummy forces. The principal of virtual work (Eq. (2.5)) now is also applied to the isothermal dummy problem, where the above thermal displacements $\vec{u}_\theta$ are used as virtual displacement field for this time. Again, the virtual deformations need not be small. Due to the properties of the Dirac delta function, this gives

$$\sum_{j=1}^{n} \vec{B}_j \cdot \vec{u}_\theta(\vec{\xi}_j) = \int_{B_0} (\tilde{F}_\mathrm{i}\tilde{S}_\mathrm{d} + \tilde{G}_{u\mathrm{d}}\tilde{S}_\mathrm{i} + \tilde{G}_{u\mathrm{d}}\tilde{S}_\mathrm{d}) : \tilde{G}_{u\theta}\,\mathrm{d}V_0 - \int_{\partial B_\mathrm{d}} \vec{f}_{0\mathrm{d}} \cdot \vec{u}_\theta\,\mathrm{d}A_0. \tag{3.3}$$

For a unit single force $\vec{B}_l = 1\vec{e}_l$ acting at the place $\vec{\xi}_l$ in the reference configuration, the left side of Eq. (3.3) becomes the component of the thermally induced displacement $\vec{u}_\theta(\vec{\xi}_l)$ in the direction of the unit vector $\vec{e}_l$:

$$\sum_{j=1}^{n} \vec{B}_j \cdot \vec{u}_\theta(\vec{\xi}_j) = \vec{e}_l \cdot \vec{u}_\theta(\vec{\xi}_l) = u_{\theta l}(\vec{\xi}_l). \tag{3.4}$$

In Eq. (3.3), again use has been made of the principle of virtual work applied to the intermediate state, and this relation has been subtracted from the work formulations for the final state.

4. St. Venant–Kirchhoff's material

The above two virtual work statements, Eqs. (3.1) and (3.3), can be connected by means of constitutive equations. In the following, a special hyperelastic relation, called the St. Venant–Kirchhoff material, is utilized. Not withstanding the fact that this constitutive model has been found to be of a restricted practical use in the large strain regime, a number of theories for beams, plates and shells have been developed for the St. Venant–Kirchhoff material in order to take into account moderately large deviations from the infinitesimal theory (see Truesdell and Noll, 1972, Section 94 for an embedding into the general theory of hyperelasticity, and Section 82A for historical remarks).

In the case of an anisotropic material under thermal loading, the St. Venant–Kirchhoff constitutive equation is written as:

$$\tilde{S}_\theta = \tilde{C}[\tilde{E}_\theta] + \theta\tilde{M}, \tag{4.1}$$

where $\tilde{C}$ denotes the symmetric fourth-order elasticity tensor, and $\tilde{M}$ is the stress-temperature tensor of the thermal coefficients. The additional Green strain tensor due to the change in temperature is

$$\tilde{E}_\theta = \tfrac{1}{2}(\tilde{F}_\mathrm{i}^\mathrm{T}\tilde{G}_{u\theta} + \tilde{G}_{u\theta}^\mathrm{T}\tilde{F}_\mathrm{i} + \tilde{G}_{u\theta}^\mathrm{T}\tilde{G}_{u\theta}), \tag{4.2}$$

which may be easily derived by subtracting the intermediate Green strain from the total one, recall Eq. (2.7) for the definition of the Green strain. We note that the standard thermoelastic law of the linear theory (see Carlson, 1972) follows by setting $\tilde{F}_\mathrm{i} = \tilde{I}$ and neglecting the last quadratic term in Eq. (4.2), and by inserting the result into Eq. (4.1).

For the dummy load problem, the same material parameters are used as in Eq. (4.1), but we set $\theta_\mathrm{d}(\vec{\xi}) = 0$:

$$\tilde{S}_\mathrm{d} = \tilde{C}[\vec{E}_\mathrm{d}]. \tag{4.3}$$

$\vec{E}_\mathrm{d}$ is defined analogous to Eq. (4.2):

$$\tilde{E}_\mathrm{d} = \tfrac{1}{2}(\tilde{F}_\mathrm{i}^\mathrm{T}\tilde{G}_{u\mathrm{d}} + \tilde{G}_{u\mathrm{d}}^\mathrm{T}\tilde{F}_\mathrm{i} + \tilde{G}_{u\mathrm{d}}^\mathrm{T}\tilde{G}_{u\mathrm{d}}). \tag{4.4}$$

In the following, we apply Eqs. (4.1)–(4.4) under two circumstances. At first, we assume the intermediate configuration to coincide with the unstressed reference configuration, such that in $\tilde{F}_\mathrm{i} = \tilde{I}$ in Eqs. (4.2) and (4.4), but we do not neglect the quadratic terms in Eqs. (4.2) and (4.4). This situation will be used to model the geometrically non-linear behavior of hyperelastic solids and structures made of a St. Venant–Kirchhoff material. The stresses $\vec{S}_\theta$ and $\vec{S}_\mathrm{d}$ then represent the total second Piola–Kirchhoff stresses, and $\tilde{E}_\theta$, $\tilde{E}_\mathrm{d}$ are the total Green strains with respect to the unstressed reference configuration.

Secondly, the constitutive equations (4.1) and (4.3) are used to describe the situation of an infinitesimal strain superimposed upon a given, intermediate state of strain in a body made of an arbitrary hyperelastic material, where the intermediate stress however is not assumed to vanish. In order to do so, we neglect the last terms in Eqs. (4.2) and (4.4), but we do not set $\tilde{F}_{\mathrm{i}}$ equal to the identity tensor. The strains $\tilde{E}_\theta$ and $\tilde{E}_{\mathrm{d}}$ then represent the linearizations of the Green strains about the intermediate configuration, and the stresses $\tilde{S}_\theta$, $\tilde{S}_{\mathrm{d}}$, are the corresponding linearizations of the 2. Piola–Kirchhoff stresses for details on the linearization about a strained state.

5. Maysel's formula extended

In a first step, Maysel's formula is derived for a non-linearly deforming St. Venant–Kirchhoff material with $\tilde{S}_{\mathrm{i}} = \tilde{0}$ and $\tilde{F}_{\mathrm{i}} = \tilde{I}$. We note that $\tilde{S}_{\mathrm{d}} : \tilde{G}_{u\theta} = \tilde{S}_{\mathrm{d}} : \tilde{G}_{u\theta}^{\mathrm{T}} = \frac{1}{2}\tilde{S}_{\mathrm{d}} : (\tilde{G}_{u\theta} + \tilde{G}_{u\theta}^{\mathrm{T}})$ due to the symmetry of the second Piola–Kirchhoff stress tensor. From the definition of the Green strain, Eq. (4.2), it follows that $\tilde{S}_{\mathrm{d}} : \tilde{G}_{u\theta} = \tilde{S}_{\mathrm{d}} : (\tilde{E}_\theta - \frac{1}{2}(\tilde{G}_{u\theta}^{\mathrm{T}}\tilde{G}_{u\theta})$. Using Eq. (4.3), the integrand of the volume integral in Eq. (3.3) thus reads

$$\left[(\tilde{I} + \tilde{G}_{u\mathrm{d}})\tilde{S}_{\mathrm{d}}\right] : \tilde{G}_{u\theta} = \tilde{C}[\tilde{E}_{\mathrm{d}}] : \tilde{E}_\theta - \tfrac{1}{2}\tilde{S}_{\mathrm{d}} : (\tilde{G}_{u\theta}^{\mathrm{T}}\tilde{G}_{u\theta}) + \tilde{S}_{\mathrm{d}} : (\tilde{G}_{u\mathrm{d}}^{\mathrm{T}}\tilde{G}_{u\theta}), \tag{5.1}$$

Analogously, one obtains for the integrand in Eq. (3.1):

$$(\tilde{F}_\theta\tilde{S}_\theta) : \tilde{G}_{u\mathrm{d}} = \tilde{C}[\tilde{E}_\theta] : \tilde{E}_{\mathrm{d}} + \theta\tilde{M} : \tilde{E}_{\mathrm{d}} - \tfrac{1}{2}\tilde{S}_\theta : (\tilde{G}_{u\mathrm{d}}^{\mathrm{T}}\tilde{G}_{u\mathrm{d}}) + \tilde{S}_\theta : (\tilde{G}_{u\mathrm{d}}^{\mathrm{T}}\tilde{G}_{u\theta}). \tag{5.2}$$

As a consequence of the symmetry of the tensor of elasticity $\tilde{C}$, the first two terms at the right-hand sides of Eqs. (5.1) and (5.2) are equal, $\tilde{C}\left[\tilde{E}_{\mathrm{d}}\right] : \tilde{E}_\theta = \tilde{C}\left[\tilde{E}_\theta\right] : \tilde{E}_{\mathrm{d}}$. Taking the elasticity tensor to be invertible, the stress-temperature tensor is replaced by the thermal expansion tensor $\tilde{A} = -\tilde{C}^{-1}\left[\tilde{M}\right]$ (see Carlson, 1972, Section 7). The virtual work statements, Eqs. (3.1) and (3.3), eventually can be combined to the relation

$$\begin{aligned} u_{\theta l}(\vec{\xi}_l) = {} & \int_{B_\theta} \operatorname{tr}(\tilde{S}_{\mathrm{d}}\tilde{A})\theta\,\mathrm{d}V_0 + \frac{1}{2}\int_{B_\theta}\left\{\tilde{S}_\theta : (\tilde{G}_{u\mathrm{d}}^{\mathrm{T}}\tilde{G}_{u\mathrm{d}}) - \tilde{S}_{\mathrm{d}} : (\tilde{G}_{u\theta}^{\mathrm{T}}\tilde{G}_{u\theta})\right\}\mathrm{d}V_0 \\ & + \int_{B_\theta}(\tilde{S}_{\mathrm{d}} - \tilde{S}_\theta) : (\tilde{G}_{u\mathrm{d}}^{\mathrm{T}}\tilde{G}_{u\theta})\,\mathrm{d}V_0 + \int_{\partial B_\theta}\vec{f}_{0\theta}\cdot\vec{u}_{\mathrm{d}}\,\mathrm{d}A_0 - \int_{\partial B_{\mathrm{d}}}\vec{f}_{0\mathrm{d}}\cdot\vec{u}_\theta\,\mathrm{d}A_0. \end{aligned} \tag{5.3}$$

The first term at the right-hand side of Eq. (5.3) corresponds to Maysel's formula of the infinitesimal theory (see Carlson, 1972). In the non-linear field theory, conduction of heat takes place in the deformed configuration of the body. Thus, in general, the temperature distribution θ is not known in advance. The expression (5.3) nevertheless holds, and it represents Maysel's formula extended to the non-linear field theory. In the range of moderately large deformations, which is the scope of the present formulation, θ may be prescribed as a function of the place $\vec{\xi}$ in the reference configuration. The non-linear integral statement of Eq. (5.3) then may be directly used for the solution of the thermal problem, once the dummy solution is known. This will be demonstrated in the next section for anisotropic, shear-deformable beams according to the v. Karman theory of moderately large strains. Note that the presence of the surface integrals in Eq. (5.3) gives the liberty to use different boundary conditions to be applied in the thermal and in the dummy problem.

In the remainder of this section, we study the problem of an infinitesimal strain superimposed upon a given, intermediate state of strain in an hyperelastic material of arbitrary type. Omitting the quadratic terms in the definition of the Green strain, Eqs. (4.2) and (4.4), it can be shown that $(\tilde{F}_{\mathrm{i}}\tilde{S}_{\mathrm{d}}) : \tilde{G}_{u\theta} = \tilde{S}_{\mathrm{d}} : \tilde{E}_\theta$, and $(\tilde{F}_{\mathrm{i}}\tilde{S}_\theta) : \tilde{G}_{u\mathrm{d}} = \tilde{S}_\theta : \tilde{E}_{\mathrm{d}}$. Since we consider the linearizations about the intermediate state, the terms $\tilde{G}_{u\theta}\tilde{S}_\theta$ and $\tilde{G}_{u\mathrm{d}}\tilde{S}_{\mathrm{d}}$ in the volume integrals of Eqs. (3.1) and (3.3) are also neglected. The integrand of the volume integral in Eq. (3.3) now reads:

$$(\tilde{F}_{\rm i}\tilde{S}_{\rm d} + \tilde{G}_{u{\rm d}}\tilde{S}_{\rm i}) : \tilde{G}_{u\theta} = \tilde{C}\left[\tilde{E}_{\rm d}\right] : \tilde{E}_\theta + \tilde{S}_{\rm i} : (\tilde{G}^{\rm T}_{u{\rm d}}\tilde{G}_{u\theta}). \tag{5.4}$$

Analogously, one obtains for the integrand of the volume integral in Eq. (3.1):

$$(\tilde{F}_{\rm i}\tilde{S}_\theta + \tilde{G}_{u\theta}\tilde{S}_{\rm i}) : \tilde{G}_{u{\rm d}} = \tilde{C}\left[\tilde{E}_\theta\right] : \tilde{E}_{\rm d} + \theta\tilde{M} : \tilde{E}_{\rm d} + \tilde{S}_{\rm i} : (\tilde{G}^{\rm T}_{u\theta}G_{u{\rm d}}). \tag{5.5}$$

The right-hand side of Eq. (5.4) can be inserted into Eq. (5.5). The virtual work statements, Eqs. (3.1) and (3.3), can thus be combined to the classical form of Maysel's equation

$$u_{\theta l}(\vec{\xi}_l) = \int_{B_\theta} \theta\,{\rm tr}(\tilde{S}_{\rm d}\tilde{A})\,{\rm d}V_0 + \int_{\partial B_\theta} \vec{f}_{0\theta}\cdot\vec{u}_{\rm d}\,{\rm d}A_0 - \int_{\partial B_{\rm d}} \vec{f}_{0{\rm d}}\cdot\vec{u}_\theta\,{\rm d}A_0, \tag{5.6}$$

see Ziegler and Irschik (1987) for the linear case of infinite strains superimposed upon the unstressed reference configuration. Hence, Maysel's formula can be applied directly to the problem of an infinitesimal thermally induced strain, superimposed upon a given arbitrary state of strain, once the corresponding dummy force solution is known. In contrast, the integral equation (5.3) for the St. Venant–Kirchhoff problem is not a linear one. Choosing a suitable dummy problem, however, it may be possible to obtain a linear integral equation also in the case of Eq. (5.3). This will be demonstrated in a subsequent structural application.

6. Maysel's formula for the v. Karman theory of orthotropic, shear-deformable beams

As a structural application of the above extension of Maysel's formula to the non-linear regime, moderately large deformations of initially straight beams are studied in the following. The coordinate of the undeformed beam axis in the reference configuration is denoted by X, where $0 \leqslant X \leqslant L_0$. The deformations are assumed to take place in the (X,Z)-plane, where Z is the transverse coordinate. u, w denote axial displacement and transverse deflection of the beam axis, respectively. Subsequently, we use the abbreviation $(\ )' = \partial/\partial X$.

The equilibrium equations, formulated by means of stress resultants, are written as:

$$N' + q_x = 0 \tag{6.1a}$$

$$Q' + (Nw')' + q_z = 0, \tag{6.1b}$$

$$M' - Q + m = 0. \tag{6.1c}$$

M, N and Q denote bending moment, axial force and transverse shear force, respectively. The last terms in the left-hand side of Eqs. (6.1a)–(6.1c) represent the imposed external loading, where q_x and q_z are forces, and m stands for a distribution of external couples. Correspondingly, the principle of virtual work reads:

$$\int_{L_0} (q_x\bar{u} + q_z\bar{w} + m\bar{\Psi})\,{\rm d}X + (N\bar{u} + (Q + Nw')\bar{w} + M\bar{\Psi})|_{X=(0,L_0)} - \int_{L_0} N(\bar{u}' + w'\bar{w}')\,{\rm d}X$$
$$- \int_{L_0} M\bar{\Psi}'\,{\rm d}X - \int_{L_0} Q(\bar{w}' + \bar{\Psi})\,{\rm d}X = 0, \tag{6.2}$$

(compare Eq. (2.5)).

The constitutive equations for axial stress and shear stress, Eqs. (4.1) and (4.3), read in case of a beam made of an orthotropic material

$$S_{xx} = \bar{C}E_{xx} + \theta\bar{M}, \quad S_{xz} = C_{3131}E_{xz}, \tag{6.3}$$

where the effective coefficients are related to the three-dimensional elasticities by

$$\bar{C} = C_{1111} - \frac{C_{3333}C_{1122}^2 - 2C_{2233}C_{1122}C_{1133} + C_{2222}C_{1133}^2}{C_{2222}C_{3333} - C_{2233}^2}$$
$$\bar{M} = M_{11} - \frac{(C_{1122}C_{3333} - C_{1133}C_{2233})M_{22} + (C_{1133}C_{2222} - C_{1122}C_{2233})M_{33}}{C_{2222}C_{3333} - C_{2233}^2}. \tag{6.4}$$

In Eqs. (6.3) and (6.4), the index 1 corresponds to the axial (X-) coordinate of the beam, and the index 3 refers to the transverse Z-coordinate. The kinematic behavior of the beam is described by combining the Timoshenko hypothesis for shear-deformable beams with the v. Karman hypothesis for moderately large deformations (see e.g. Ziegler, 1995) for these theories. Formulation of the constitutive equations at the level of stress resultants thus reads:

$$N = D(u' + \tfrac{1}{2}w'^2 - \bar{E}_{xx}^0), \tag{6.5a}$$
$$Q = S(\Psi + w'), \tag{6.5b}$$
$$M = K(\Psi' - \bar{\chi}^0), \tag{6.5c}$$

where A denotes the cross-section of the beam in the reference configuration, and $D = (\bar{C}A)_{\text{eff}}$, $S = \kappa(C_{3131}A)_{\text{eff}}$ and $K = (\bar{C}I_y)_{\text{eff}}$ are (effective) extensional, shear and bending stiffness, respectively, following from an integration over the (possibly) inhomogeneous cross-section. The shear factor is denoted by κ. Thermal loading in Eqs. (6.5a)–(6.5c) is characterized by the cross-sectional integrals

$$\bar{E}_{xx}^0 = \frac{1}{D}\int_A \theta\bar{M}\,\mathrm{d}A, \quad \bar{\chi}^0 = -\frac{1}{K}\int_A \theta\bar{M}Z\,\mathrm{d}A, \tag{6.6}$$

where $\bar{E}_{xx}^0$ is the actuating mean thermal strain, and $\bar{\chi}^0$ is the actuating thermal curvature, not to be confused with thermally induced strains and curvatures.

Running through the procedure described above for three-dimensional anisotropic solids, we end with the beam-type version of Maysel's formula for orthotropic Timoshenko beams, extended to v. Karman-type non-linear strains:

$$\begin{aligned} F_x u_\theta(\xi) + F_z w_\theta(\xi) = {} & \int_{L_0} (N_\mathrm{d}\bar{E}_{xx}^0 + M_\mathrm{d}\bar{\chi}^0)\,\mathrm{d}X + \frac{1}{2}\int_{L_0} (N_\theta w_\mathrm{d}'^2 - N_\mathrm{d} w_\theta'^2)\,\mathrm{d}X \\ & + \int_{L_0} (N_\mathrm{d} - N_\theta) w_\mathrm{d}' w_\theta'\,\mathrm{d}X + (N_\mathrm{d} u_\theta + (Q_\mathrm{d} + N_\mathrm{d} w_\mathrm{d}')w_\theta + M_\mathrm{d}\Psi_\theta)|_{X=(0,L_0)} \\ & - (N_\theta u_\mathrm{d} + (Q_\theta + N_\theta w_\theta')w_\mathrm{d} + M_\theta \Psi_\mathrm{d})|_{X=(0,L_0)}. \end{aligned} \tag{6.7}$$

Note the obvious similarity between Eq. (6.7) and the three-dimensional formulation Eq. (5.3). The boundary terms in Eq. (6.7) vanish, if homogeneous boundary conditions are prescribed.

7. Thermally loaded simply supported beam with axially fixed supports

As a benchmark problem, a beam simply supported at $X = 0$ and $X = L_0$ is studied, where the axial displacements of both ends are assumed to be prevented. The beam is subjected to a thermal curvature $\bar{\chi}^0 = \text{const.}$ and mean temperature $\bar{E}_{xx}^0 = \text{const.}$ In the geometrically non-linear range of the v. Karman theory, an axial force N_θ is produced due to this type of thermal loading in the beam with fixed ends. The deflection is calculated using Maysel's formula, Eq. (6.7). We are especially interested in the behavior of the beam in the post-buckling range.

For convenience, a simplified dummy problem is considered. The transverse dummy single force $F = 1$ is applied at $X = \xi$, and we allow an axial displacement of the support at $X = L_0$ for the dummy problem. No dummy axial force is applied at this end. Hence, the dummy normal force vanishes throughout the beam, $N_{\mathrm{d}}(X) = 0$, see Eq. (6.1a). After span-wise integration, Eq. (6.5a) then leads to

$$u_{\mathrm{d}}(X = L_0) = -\tfrac{1}{2}\int_{L_0} w_{\mathrm{d}}'^2 \,\mathrm{d}X. \tag{7.1}$$

Maysel's formula, Eq. (6.7), thus becomes a formally linear integral equation for the thermal deflection:

$$w_\theta(\xi) = \int_{L_0} M_{\mathrm{d}}\bar{\chi}^0 \,\mathrm{d}X - N_\theta \int_{L_0} w_{\mathrm{d}}'' w_\theta \,\mathrm{d}X, \tag{7.2}$$

where we have taken into account the homogeneous conditions at the ends of the beam. Integration by parts has been used in Eq. (7.2), and the thermal normal force has been set constant according to Eq. (6.1a). Eq. (7.2) is accompanied by Eq. (6.5a), integrated over the beam axis in the reference configuration:

$$N_\theta = \frac{D}{2L}\int_{L_0} w_\theta'^2 \,\mathrm{d}X - D\bar{E}_{xx}^0. \tag{7.3}$$

Eqs. (7.2) and (7.3) form the problem to be solved. Assuming N_θ to be known, Eq. (7.1) can be formally written down according to the theory of linear integral equations. Putting the result into Eq. (7.3), a non-linear algebraic equation is obtained for N_θ.

For a solution of Eq. (7.2), it is instructive to start with the deflection of the dummy problem, however assuming the beam to be rigid in shear:

$$\begin{aligned} w_{\mathrm{d\,BE}}(X) &= -\frac{1}{6}\frac{X\left(-X^2L_0 + X^2\xi - 3\xi^2 L_0 + 2\xi L_0^2 + \xi^3\right)}{L_0 K} \quad 0 \leqslant X \leqslant \xi, \\ w_{\mathrm{d\,BE}}(X) &= -\frac{1}{6}\frac{\xi\left(-\xi^2 L_0 + \xi^2 X - 3X^2 L_0 + 2XL_0^2 + X^3\right)}{L_0 K} \quad \xi \leqslant X \leqslant L_0, \end{aligned} \tag{7.4}$$

where the index BE refers to the Bernoulli–Euler theory of beams rigid in shear. Since the dummy problem is statically determinate, there is

$$M_{\mathrm{d}} = M_{\mathrm{d\,BE}} = -Kw_{\mathrm{d\,BE}}''. \tag{7.5}$$

Furthermore, it can be shown that the dummy Timoshenko deflection is related to its Bernoulli–Euler counterpart by

$$w_{\mathrm{d}}'' = w_{\mathrm{d\,BE}}'' - \frac{1}{S}\delta(X - \xi). \tag{7.6}$$

This follows from Eq. (7.5), inserted into Eqs. (6.5b), (6.5c) and (6.1b), which have to be applied to the dummy problem with transverse loading , $q_{z\mathrm{d}} = 1\delta(X - \xi)$, compare Eq. (3.2). Inserting Eqs. (7.5) and (7.6) into Eq. (7.2), and utilizing the properties of Dirac's function δ, we obtain

$$w_\theta\left(1 + \frac{N_\theta}{S}\right) = \bar{\chi}_y^0 \int_{L_0} M_{\mathrm{d\,BE}} \,\mathrm{d}X + N_\theta \int_{L_0} w_{\mathrm{d\,BE}}'' w_\theta \,\mathrm{d}X. \tag{7.7}$$

Quite naturally, the characteristic number

$$\gamma = \frac{1}{1 + \frac{N_\theta}{S}} \tag{7.8}$$

thus comes into the play, where $\gamma = 1$ for the Bernoulli–Euler theory of beams rigid in shear. For a solution of Eq. (7.7), we set

$$w_\theta = A\sin\left(\hat{k}x\right) + B\cos\left(\hat{k}x\right) + C. \tag{7.9}$$

Inserting and taking into account the boundary conditions gives

$$w_\theta = \gamma\frac{1}{\hat{k}^2}\bar{\chi}_y^0\left(1 - \cos\left(\hat{k}X\right) + \frac{\cos\left(\hat{k}L_0\right) - 1}{\sin\left(\hat{k}L_0\right)}\sin\left(\hat{k}X\right)\right), \tag{7.10}$$

with the characteristic number

$$\hat{k} = \sqrt{\frac{\gamma N}{K}}. \tag{7.11}$$

Eq. (7.10) eventually is inserted into Eq. (7.3), which then is solved numerically.

8. Numerical results

The semi-analytic solution procedure of Section 7 has been implemented in a MAPLE V.5-code, and the outcome has been compared to finite element computations using the numerical code ABAQUS, version 8.5.14. For numerical calculations, we have used a beam of span $L_0 = 1$ m, and a quadratic cross-section with height $H_0 = 0.05$ m has been considered. The axial elasticity parameter has been set to $\bar{C} = 2.1 \times 10^{11}$ N/m^2, and the thermal expansion coefficient was $\bar{M} = 1.2 \times 10^{-6}$ K^{-1}. In the isotropic case, there is $\bar{C}/C_{3131} = 2.6$. In order to model orthotropic beams made of pyrolitic-graphite type materials, much larger values have been considered in the present study, where we used a range of $\bar{C}/C_{3131} = 20$–50 (see Wu and Vinson, 1969). In the following, numerical results are presented in non-dimensional form, where the abbreviations

$$w^* = w_\theta\left(X = \frac{L_0}{2}\right)/H_0, \quad \varepsilon^* = \frac{\bar{E}^0_{xx}}{\bar{E}^0_{xx_{\mathrm{crit}}}}, \quad \kappa^* = \bar{\chi}_y^0 H_0, \quad N^* = \frac{N_\theta}{N_{\mathrm{crit}}} \tag{8.1}$$

are used. Note that

$$N_{\mathrm{crit}} = \frac{\pi^2 K}{L_0^2}, \quad \bar{E}^0_{xx_{\mathrm{crit}}} = \frac{N_{\mathrm{crit}}}{D} \tag{8.2}$$

represent the Euler buckling force and the critical mean temperature, respectively, of an isotropic beam according to the Bernoulli–Euler theory. For convenience, compressive normal forces now are taken positive.

Figs. 1 and 2 show the non-dimensional normal force N^* and the mid-span deflection w^*, as obtained by the present procedure using MAPLE V.5. As a non-dimensionalized imposed thermal curvature, the value $\kappa^* = 2.4 \times 10^{-5}$ has been used. Note the strong dependency of the solutions given in Figs. 1 and 2 from the anisotropy-parameter $\bar{C}/C_{3131}$. Above the respective critical value of ε^*, in the post-buckling range, three branches of the solutions are shown in Figs. 1 and 2. For isotropic beams rigid in shear, this fact has been pointed out by Alblas (1969), where a single-term Ritz–Ansatz has been used. A multi-mode approximation has been presented by Irschik (1986) for simply supported, isotropic thin plates. The present contribution represents an extension with respect to the use of closed form solutions for the deflection, shear-deformability and orthotropic material behavior. Following the arguments given by Irschik (1986), the outer branches of the solutions presented in Figs. 1 and 2 are stable solutions, and the middle curve represents an unstable solution.

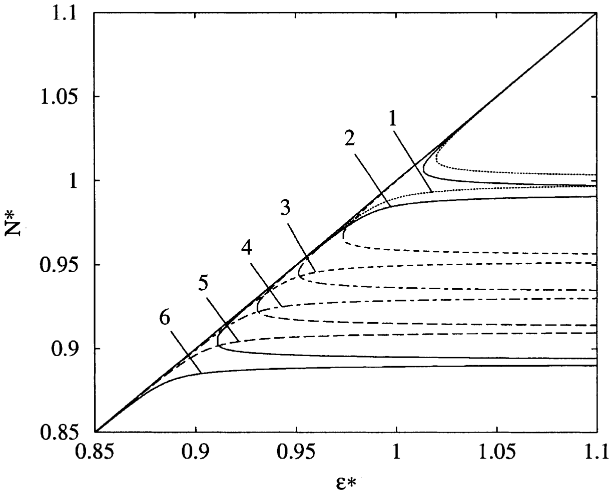

Fig. 1. Normalized normal force $N^* = N_0/N_{0_{crit}}$ for $\kappa^* = 2.4 \times 10^{-5}$ and different material parameters: (1) Bernoulli–Euler theory, isotropic material; (2) Timoshenko theory, isotropic material; (3)–(6) Timoshenko theory, $\bar{C}/C_{3131} = 20$–50.

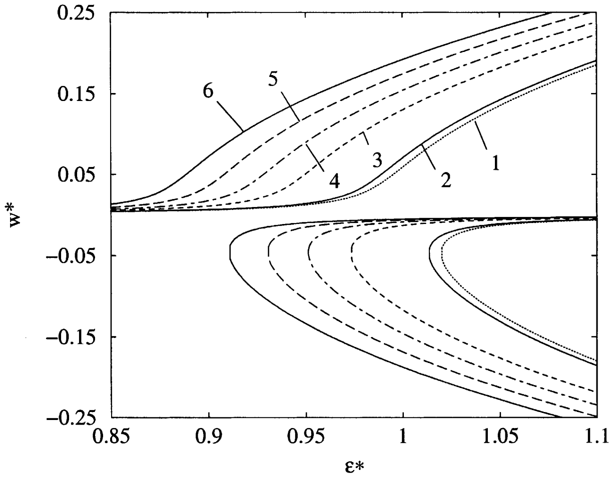

Fig. 2. Normalized mid-span deflection w^* for $\kappa^* = 2.4 \times 10^{-5}$ and different material parameters: (1) Bernoulli–Euler theory, isotropic material; (2) Timoshenko theory, isotropic material; (3)–(6) Timoshenko theory, $\bar{C}/C_{3131} = 20$–50.

Figs. 1 and 2 have been compared to the results of the finite element code ABAQUS, version 8.5.14. The beam has been subdivided into 90 isotropic elements of the type B12. In our hands, ABAQUS first showed a paradoxical behavior, represented in Fig. 3, where we used the standard incrementation step of ABAQUS, allowing a maximum increment of 0.001 for a quasi-time period of 1. In this procedure, each of the solutions depicted in Fig. 3 is obtained separately, starting from the unloaded position of the beam. Above the critical value of ε^*, ABAQUS comes out with the solutions located at the unstable branch in Fig. 2, leading to an apparent jump in Fig. 3.

A full set of solutions is shown in Fig. 4 for two values of κ^*, namely $\kappa^* = 1.2 \times 10^{-6}$ and $\kappa^* = 1.2 \times 10^{-4}$. To reach the different branches of Fig. 4, we use the standard step size for the branch already shown in Fig. 3. In order to reach the lower branches, we perform two steps. In a first step we

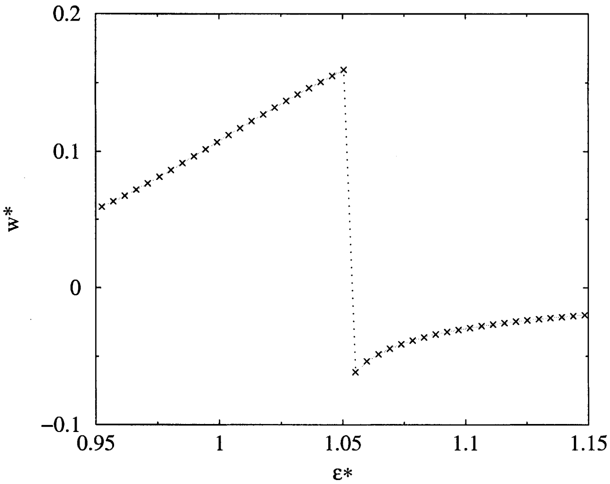

Fig. 3. Output of ABAQUS when increasing the mean temperature for a shear-deformable beam made of isotropic material, $\kappa^* = 1.2 \times 10^{-4}$.

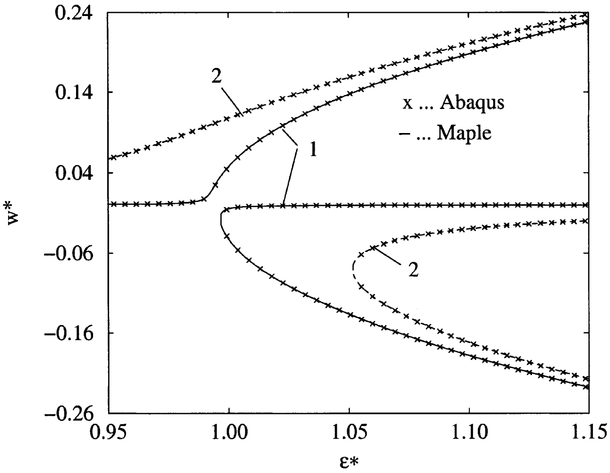

Fig. 4. Comparison between FE calculation by ABAQUS and analytical solution obtained by MAPLE V.5: (1) $\kappa^* = 1.2 \times 10^{-6}$, (2) $\kappa^* = 1.2 \times 10^{-4}$.

pre-bend the beam with $\kappa^* = -0.24$. This solution forms the initial conditions for the second step, where the actual thermal loading is applied using default incrementation.

What remains, is to prove the stability of the three solution branches in Figs. 2 and 4. In our study, this has been performed by means of a dynamic ABAQUS computation, where a small transverse force has been superimposed upon the thermoelastic static deflection of the isotropic beam. A box-type time evolution of the transverse force has been used. When starting from a stable branch, the superimposed dynamic solution should vibrate about this thermoelastic solution and should die out due to damping. This behavior indeed has been found for the outer branches in Figs. 2 and 4. This is exemplary demonstrated in Fig. 5, where the time evolution of the superimposed vibration about the outer branches of the thermoelastic solution is drawn within the stability chart for $\kappa^* = 1.2 \times 10^{-4}$. Contrary, when starting from the middle branch of the

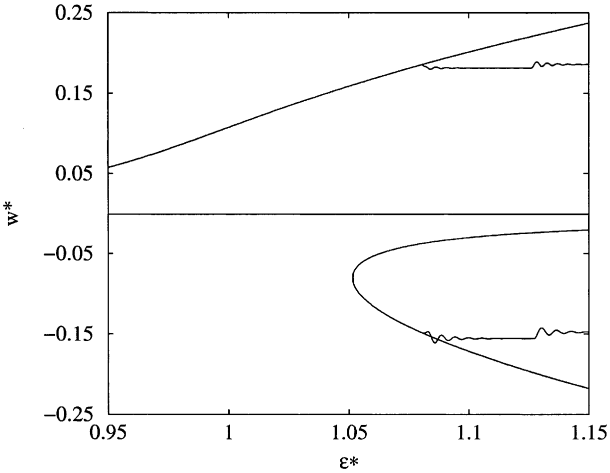

Fig. 5. Stable solution paths with a superimposed dynamic solution according to ABAQUS, shear-deformable beam made of isotropic material, $\kappa^* = 1.2 \times 10^{-4}$.

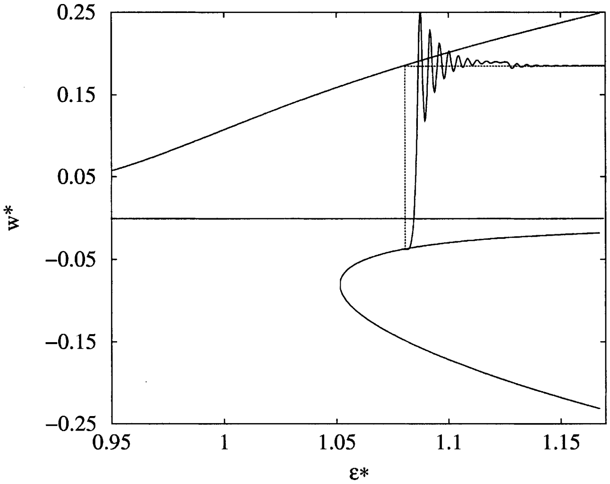

Fig. 6. Unstable solution path with a superimposed dynamic solution according to ABAQUS, shear-deformable beam made of isotropic material, $\kappa^* = 1.2 \times 10^{-4}$.

stability chart, the superimposed dynamic solution immediately jumps to the upper (stable) branch (see Fig. 6). Instability of the middle thermoelastic solution branches in Figs. 2 and 4 is thus demonstrated. Since, however, the standard incrementation procedure of ABAQUS may come out with this unstable solution, compare Fig. 3, it is recommended to superimpose a small dynamic loading for checking stability, especially in the case of more complex structures.

9. Conclusion

Maysel's formula of the linear theory of thermoelasticity has been extended to the non-linear regime. It has been shown that the classical form of this formula can be directly applied to the case of an infinitesimal strain superimposed upon a given state of large strain in an anisotropic hyperelastic body. Furthermore, an extension of Maysel's formula has been presented for the case of large strains in an anisotropic body made of a St. Venant–Kirchhoff material. A counterpart of this extended Maysel's formula has been presented for the case of a shear-deformable beam exhibiting moderately large strains according to the theory of v. Karman. A numerical study has been performed for the post-buckling regime of thermally loaded orthotropic beams made of a pyrolitic–graphite type material. Our paper has been also motivated by an analogy between thermal strains and incompatible parts of strain, such as viscoplastic or piezoelectric parts of strain. The presented results should find further practical applications in the context of these analogies. A corresponding study is presently under investigation.

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